Direction–Direction Correlations of Oriented Polymers

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We calculate the direction–direction correlations between the tangent vectors of an oriented self-avoiding walk (SAW). Let $J^\mu(x)$ and $J^\nu(0)$ be components of unit-length tangent vectors of an oriented SAW, at the spatial points x and 0, respectively. Then for distances $|x|$ much less than the average distance between the endpoints of the walk, the correlation function of $J^\mu(x)$ with $J^\nu(0)$ has, in d dimensions, the form $\langle J^\mu(x) J^\nu(0) \rangle = k(d)(x^\mu x^\nu - \frac{1}{2} x^2 \delta^{\mu\nu})/|x|^{2d}$. The dimensionless amplitude $k(d)$ is universal, and can be calculated exactly in two dimensions by using Coulomb gas techniques, where it is found to be $k(2) = 12/25\pi^2$. In three dimensions, the ε-expansion to second order in ε together with the exact value of $k(2)$ in two dimensions allows the estimate $k(3) = 0.0178 \pm 0.0005$. In dimensions $d \geqslant 4$, the universal amplitude $k(d)$ of the direction–direction correlation functions of an oriented SAW is the same as the universal amplitude of the direction–direction correlation functions of an oriented random walk, and is given by $k(d) = \Gamma^2(d/2)/(d-2)\pi^d$.

1. INTRODUCTION

A single long polymer in good solution is a critical system (see, e.g., ref. 1). The inverse length l^{-1} of the polymer plays the role of reduced temperature $T - T_c$, and the critical point is at $l = \infty$. As $l \to \infty$, certain geometrical properties of the polymer scale in a simple way with l. For instance, the average spatial distance R between the endpoints of a linear polymer is given by

$$R \sim a l^\nu, \qquad l \to \infty \tag{1}$$

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where a is independent of l. The exponents (e.g., v) and dimensionless ratios of amplitudes of these scaling laws do not depend on the detailed molecular structure of the polymer: they are universal. To calculate these universal quantities, we may work with the simplest system that behaves like a polymer in good solution. The relevant properties are (1) flexibility (a polymer assumes its configurations with equal probability) and (2) excluded volume (the short-range repulsion between a polymer's constituent monomers prevents it from intersecting itself). The simplest object which possesses these properties is a self-avoiding walk, i.e., a walk on a lattice, which never visits the same site more than once. Therefore, in order to calculate universal quantities characterizing the asymptotic geometry of a polymer, we may restrict our attention to SAWs.

In this article, we shall study oriented SAWs (polymers). An oriented SAW is simply a SAW with an "arrow" that runs along its length (Figs. 1 and 2). A polymer, for example, with the structure $\cdots$-A-B-C-A-B-C-A-$\cdots$ is oriented, because the sequence defined by A $\rightarrow$ B $\rightarrow$ C is distinct from the defined by A $\rightarrow$ C $\rightarrow$ B. Alternatively, a polymer could be composed of monomers with dipole moments joined "head to tail." We shall take the probability distribution of spatial configurations of an oriented SAW to be the same as that of a SAW without orientation. That is, we take all allowed spatial configurations of an oriented SAW of fixed length to be equally probable. The average shape of a long oriented SAW is therefore the same as the average shape of a SAW without orientation; e.g., they both have the same exponent v. However, an oriented SAW has, in addition, a direction. The direction of a given step of an oriented SAW can be represented by a vector tangent to the step, with the arrow of the tangent vector pointing in the same direction as the arrow on the walk. We
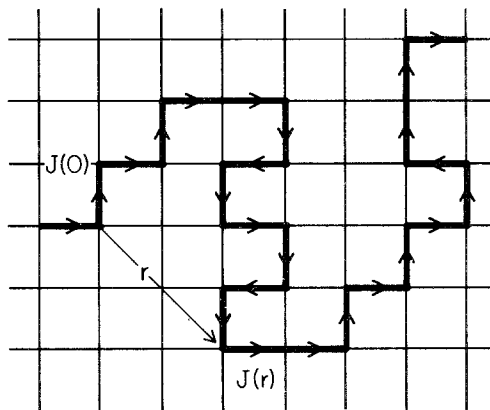

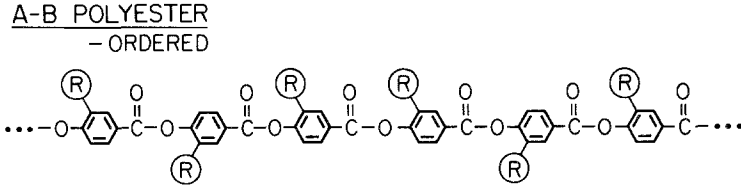Fig. 1. An oriented SAW on a two-dimensional square lattice.

A-B POLYESTER
— ORDERED

Fig. 2.   A–B polyester, an oriented polymer.

shall calculate the correlation functions of these tangent vectors $J(x)$ at different points along an oriented SAW. In particular, we shall calculate the direction–direction correlation functions $\langle J^{\mu}(x) J^{\nu}(0) \rangle$ for distances $|x|$ between the tangent vectors J whose correlations we are considering much less than the average distance R between the endpoints of the walk, but much greater than the lattice spacing. This is the critical regime, where we expect universal behavior. It should be noted that the vector x is the position of a step of the walk in space, and not the distance along the length of the walk.

There are essentially three steps in the calculation. We first generalize a remarkable correspondence, found by de Gennes,[2] between SAWs and an $O(N)$ model, to oriented SAWs and a complex $O(N)$ model. More precisely, wed show that oriented SAWs are described by the $N \to 0$ limit of the complex $O(N)$ model, and identify on the lattice the operator in the complex $O(N)$ model which corresponds to the tangent vector of an oriented SAW. It turns out to be a conserved current, associated with a $U(1)$ symmetry of the complex $O(N)$ model. The critical behavior of the current–current correlation functions of the $N \to 0$ complex $O(N)$ model thus gives us the direction–direction correlation functions of an oriented SAW, on distance scales much less than the average distance between the endpoints of the walk (the "correlation length"). We then show that, at the critical point of the complex $O(N)$ model, general considerations completely determine the functional form of the current–current correlation functions. These considerations are rotational covariance, current conservation, and dimensional analysis. We can use dimensional analysis to understand how the current–current correlation functions scale under dilatations because conserved currents associated with internal symmetries of a theory do not acquire an anomalous dimension.[3] Finally, we calculate the dimensionless amplitude $k(d)$. Note that since $k(d)$ is dimensionless, it is universal (once the normalization of J has been fixed; in Section 4 we shall show that a natural normalization exists).

In the calculation of the universal amplitude $k(d)$ one may distinguish three cases depending on the dimension d of space. In two dimensions $k(2)$ can be calculated exactly by mapping the complex $O(N)$ model onto

a Coulomb gas.[4,5] In dimensions $d \geqslant 4$, oriented SAWs are correctly described by oriented random walks, and these in turn are described by a complex Gaussian model.[6] The Gaussian model is a free field theory, so $k(d)$ for $d \geqslant 4$ can be calculated exactly by using Wick's theorem.[7] In dimensions $2 < d < 4$ we shall apply the ε-expansion[8,9] to the continuum $O(N)$ model to compute the amplitude $k(d)$ to second order in ε. This, along with the exact value of $k(2)$, gives us an estimate for $k(3)$.

It should be possible to measure experimentally the amplitude $k(d)$ of the direction–direction correlation functions of an oriented polymer. In practice, however, it may be difficult to distinguish one orientation of an oriented monomer from the opposite orientation.

This paper has the following outline. In Section 2 we define the complex $O(N)$ model on a hypercubic lattice, and show that the tangent vectors J of an oriented SAW correspond to a conserved current in the complex $O(N)$ model. In Section 3, we use this correspondence to calculate the form of the direction–direction correlation functions, and to calculate in the continuum complex Gaussian model the amplitude $k_G(d)$ for oriented random walks. We also discuss what the direction–direction correlation functions look like. In Section 4 we fix the normalization of J. In Section 5, $k(2)$ is calculated exactly in two dimensions by mapping the $O(N)$ model onto a Coulomb gas. In Section 6, $k(d)$ is calculated to second order in the ε-expansion. The Feynman diagrams which contribute to the ε-expansion are evaluated in the appendices.

2. ORIENTED SAWs AND THE COMPLEX $O(N)$ MODEL

In this section we will work on a d-dimensional hypercubic lattice generated by the orthonormal vectors $\hat{e}_\mu$. Vertices will be labeled by vectors x without indices, and lattice edges by vectors with indices, so that x^μ labels the lattice edge between the vertex x and $x + \hat{e}_\mu$. Likewise, a vector lying on the lattice edge x^μ is written $J^\mu(x)$.

The complex $O(N)$ model on the lattice has N complex-valued spins $S_i(x)$, $S_i^*(x)$ at each vertex, and lattice Hamiltonian

$$\beta H = -\beta \sum_{\langle x, x' \rangle} \sum_{i=1}^{N} S_i(x) S_i^*(x') + S_i^*(x) S_i(x') \tag{2}$$

The first sum runs over all nearest neighbor pairs of the lattice, and the spins are normalized so that $\sum_{i=1}^{N} S_i S_i^* = N$.

The connection between the complex $O(N)$ model and oriented SAWs follows from a graphical representation of the high-temperature expansion

of the complex $O(N)$ model. This is an expansion of the complex $O(N)$ partition function or correlation functions in powers of β. To be definite, consider

$$Z = \text{Tr } e^{-\beta H} = \int \prod_x d\Omega_x \prod_{\langle x, x' \rangle} \prod_{i=1}^{N} e^{\beta(S_i(x) S_i^*(x') + S_i^*(x) S_i(x'))} \qquad (3)$$

Here $d\Omega_x$ is the normalized angular measure $\int d\Omega_x = 1$ of the $(2N-1)$-sphere defined by $\sum_{i=1}^{N} S_i S_i^* = N$. Now imagine expanding the integrand in powers of β. Each term in the expansion is the trace of products of spins $S_i(x) S_i^*(x')$ and $S_i^*(x) S_i(x')$ times β raised to some power. The graphical representation consists in assigning to each term in the expansion a graph on the lattice, with an oreinted bond pointing from x' to x, for each factor of $S_i(x) S_i^*(x')$, and an oriented bond pointing in the opposite direction, from x to x', for each factor of $S_i^*(x) S_i(x')$.

In the limit $N \to 0$, only two kinds of terms are nonzero: terms with no factors of a spin $S_i(x)$, and terms with a single factor of $S_i(x) S_i^*(x)$:

$$\text{Tr } S_i(x) S_j^*(x) = \prod_{x'} \int d\Omega_{x'} \, S_i(x) S_j^*(x) = \delta_{i,j} \qquad (4)$$

All higher moments are identically zero.[2,10] [The proof is the same as for the real $O(N)$ model, because the complex $O(N)$ model expressed in terms of the real and imaginary parts of the complex spins $S_i = (1/\sqrt{2})(S_{i1} + iS_{i2})$ is just a real $O(2N)$ model.] Thus the graphical expansion of the partition function is a sum over all configurations of oriented loops, each loop weighted by β^l. However, when we sum over the spin indices, each loop acquires a factor of N, which we have set to 0. Thus the only contribution to the partition function is from the graph with no loops, so that $Z = 1$ for $N = 0$.

Now consider the quantity

$$\langle S_1^*(x) \, \partial^\mu S_1(x) - S_1(x) \, \partial^\mu S_1^*(x) \rangle$$
$$= \langle S_1^*(x) \, S_1(x + \hat{e}_\mu) - S_1(x) \, S_1^*(x + \hat{e}_\mu) \rangle \qquad (5)$$

where we have used the definition of the lattice derivative as a discrete difference

$$\partial^\mu f(x) = f(x + \hat{e}_\mu) - f(x) \qquad (6)$$

From above, the diagrams which contribute to the first of the two terms of Eq. (5) are, in the $N \to 0$ limit, all oriented loops containing an oriented bond on the lattice edge x^μ pointing from x to $x + \hat{e}_\mu$ (Fig. 3). Each such
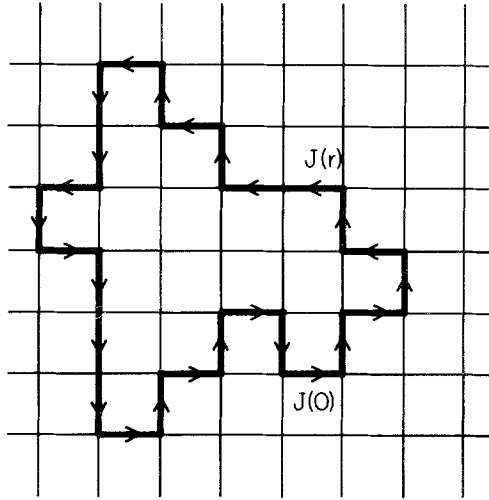
Fig. 3. A graph which contributes to $\langle J^1(r)\, J^1(0)\rangle$. Here, $J^1(0)=1$, $J^1(r)=-1$, and the graph has weight $\beta^{24.}$

loop is weighted by a fugacity β^l. On the other hand, the diagrams which contribute to the second term are all oriented loops containing an oriented bond at x^μ pointing from $x+\hat{e}_\mu$ to x; that is, with orientation opposite the loops from the first term. The difference of the first and second terms is thus

$$\sum_{l=0}^{\infty} \left[w_l(+x^\mu) - w_l(-x^\mu) \right] \beta^l \tag{7}$$

where $\omega_l(+x^\mu)$ is the number of oriented loops of length l containing an oriented bond on the lattice edge x^μ, pointing from x to $x+\hat{e}_\mu$, and $w_l(-x^\mu)$ is the number of oriented loops of length l containing an oriented bond at x^μ, but pointing in the opposite direction, from $x+\hat{e}_\mu$ to x. If we define the tangent vector $J^\mu(x)$ of an oriented loop to equal $+1$ if the loop passes through the link x^μ pointing from x to $x+\hat{e}_\mu$, -1 if pointing from $x+\hat{e}_\mu$ to x, and zero if it does not pass through the link x^μ at all, then we see that Eq. (7) is the average value of $J^\mu(x)$, in an ensemble of oriented loops, whose fluctuating lengths are weighted by a fugacity β^l. Thus we have identified the operator in the complex $O(N)$ model which represents the tangent vectors of oriented loops

$$J^\mu(x) = S_1^*(x)\, \partial^\mu S_1(x) - S_1(x)\, \partial^\mu S_1^*(x) \tag{8}$$

Of course $\langle J^\mu(x)\rangle = 0$, since for every oriented loop passing through x^μ in one direction, there is another loop, with the opposite orientation, passing

through the link in the opposite direction. However, correlations between tangent vectors at different points along an oriented loop are not zero, and it is these that we shall calculate.

It may be useful, before proceeding with the continuum calculation, to see explicitly which diagrams contribute to the correlation function of two tangent vectors on the lattice

$$\langle J^\mu(x)\, J^\nu(0)\rangle = \langle S_1^*(x)\, S_1(x+\hat{e}_\mu)\, S_1^*(0)\, S_1(\hat{e}_\nu)\rangle$$
$$- \langle S_1(x)\, S_1^*(x+\hat{e}_\mu)\, S_1^*(0)\, S_1(\hat{e}_\nu)\rangle + \text{c.c.} \qquad (9)$$

where the limit $N \to 0$ is understood. Consider the first of the four terms on the right-hand side. We have two ways to "contract" or pair a spin with a complex conjugate spin. On the one hand, we may contract $S_1^*(x)$ with $S_1(x+\hat{e}_\mu)$ and $S_1^*(0)$ with $S_1(\hat{e}_\nu)$ (contractions "at the same point"). The diagrams which contribute to this contraction are pairs of oriented loops, one containing an oriented bond at x^μ pointing from x to $x+\hat{e}_\mu$, and the other an oriented bond at 0^ν pointing from 0 to $\hat{e}_\nu$. However, diagrams of this type are exactly canceled by similar diagrams from the second term on the right-hand side of Eq. (9) (the orientation of one of the two loops in the second term is opposite the orientation of the corresponding loop in the first term, but the numerical contribution of a loop to the high-temperature expansion is independent of its orientation).

The other type of contraction pairs $S_1^*(x)$ with $S_1(\hat{e}_\nu)$ and $S_1(x+\hat{e}_\mu)$ with $S_1^*(0)$. The diagrams which contribute to this contraction are all oriented loops containing both an oriented bond at x^μ pointing from x to $x+\hat{e}_\mu$, and an oriented bond at 0^ν pointing from 0 to $\hat{e}_\nu$ (Fig. 3). The corresponding contraction in the second term on the right-hand side of Eq. (9) is given by the sum of all oriented loops which, just as in the first term, contain an oriented bond at 0^ν pointing from 0 to $\hat{e}_\nu$, but, opposite to the first term, contain an oriented bond at x^μ pointing from $x+\hat{e}_\mu$ to x. Thus the difference of the first and second terms is the average value (in our ensemble of oriented loops) of the tangent vector (which we have normalized to one) at x^μ of oriented loops which pass through the lattice edge 0^ν pointing from 0 to $\hat{e}_\nu$.

The analysis of the complex conjugate of the above two terms (the third and fourth terms on the right-hand side of Eq. (9)) is the same. Their difference is the average value of the tangent vector of an oriented loop on the link x^μ given that the loop contains an oriented bond at 0^ν pointing from $\hat{e}_\nu$ to 0. The contribution to the graphical expansion of the third and fourth terms equals the contribution from the first two terms, because, aside from having opposite orientation, exactly the same loops contribute to each pair. We may therefore regard the correlation function

$\langle J^\mu(x)\, J^\nu(0)\rangle$ as twice the average value of a tangent vector to an oriented loop at x^μ given that the loop passes through the link 0^ν in the direction 0 to $\hat{e}_\nu$.

Four things should be noted. First, the tangent vectors introduced above are proportional to one of the conserved currents associated with the $U(1)$ symmetry transformations $S_i \to e^{-i\alpha q}S_i$ and $S_i^* \to e^{i\alpha q}S_i^*$ of the complex $O(N)$ model. Here q is the $U(1)$ charge carried by the spin S. Second, the configurations which contribute to $\langle J^\mu(x)\, J^\nu(0)\rangle$ are oriented loops, not oriented SAWs. However, we are interested in the limit $l \to \infty$, for fixed distance x between segments of the oriented loop, or, for fixed l, in distances $|x|$ much less than the loop radius of gyration R_G: $|x| \ll R_G \sim l^\nu$. In this regime an oriented loop looks the same as an oriented SAW. Therefore J as defined above is also the tangent vector of an oriented SAW. Third, for a fixed value of β, only loops of length $l \sim 1/(\beta_c - \beta)$, where β_c is the inverse critical temperature of the complex $O(N)$ model, contribute appreciably to the sum. So, even though the ensemble consists of loops of fluctuating length, the calculation is valid for a SAW of fixed length. Fourth, the lattice tangent vectors J as we have defined them have a well-defined continuum limit. $J(x)$ in the continuum limit does not, however, correspond to a derivative at x with respect to a parameter (e.g., arclength), since for a given continuum SAW this does not, in general, exist.

3. FORM OF THE CORRELATION FUNCTION

In the previous section we showed that the tangent vectors of an oriented SAW are described by a $U(1)$ conserved current J of the $N \to 0$ complex $O(N)$ model. In the present section we shall show that general properties of the current J completely determine the functional form of the direction–direction correlation functions in the region of interest (distances much less than the average distance between the endpoints of the SAW, but much greater than the lattice spacing). We shall also calculate the universal amplitude $k_G(d)$ of the direction–direction correlation functions of oriented random walks.

Since we are interested in universal quantities, we may work in the continuum theory. Due to the constraint $\sum_{i=1}^{N} S_i S_i^* = N$, the continuum limit of the lattice complex $O(N)$ model is a complex nonlinear sigma model. This model in turn is believed to be in the same universality class as the continuum complex $O(N)$ model,[11] with bare action

$$S = \int d^d x \sum_{i=1}^{N} \partial^\mu S_i^* \partial_\mu S_i + \frac{g}{4}\left(\sum_{i=1}^{N} S_i S_i^*\right)^2 \tag{10}$$

The normal ordered conserved currents are $J_i^\mu = \lambda :S_i^* \partial^\mu S_i - S_i \partial^\mu S_i^*:$. The constant λ will be determined in the next section, where we normalize $J \equiv J_1$. We are interested in the critical behavior of this model, which occurs as the renormalized mass tends to zero. In dimensional regularization (which we will use), the renormalized mass is zero if the bare mass is zero, so we have set the bare mass to zero. The currents have been normal ordered because, as we saw in the lattice version of the theory, contractions at the same point cancel identically. Henceforth J will be taken to be normal ordered.

The functional form of the current–current (or direction–direction) correlation functions follows (in the region of interest) directly from rotational covariance of the correlation functions, current conservation, and dimensional analysis. Dimensional analysis determines how the correlation functions transform under changes of scale. One may wonder, "Why is it legitimate to use dimensional analysis to determine how the correlation functions scale under dilatations?" For distances much less than its average radius of gyration, the statistics of an oriented polymer is governed by the critical point of an interacting theory, a complex $O(N)$ model. One might, therefore, expect the conserved current J to acquire an anomalous dimension, and to transform under dilatations with a different power than that indicated by naive dimensional analysis. However, this is not the case: conserved currents arising from internal symmetries of a renormalizable field theory do not acquire an anomalous dimension. This follows immediately from the remarkable fact that such currents are not renormalized.[3] In the context of continuum renormalized perturbation theory this means that the multiplicative factor Z_J, which relates the renormalized current J_R to the bare current J via $J_R = Z_J^{-1} J$, can be chosen to be identically equal to one. Or, what is the same, correlation functions of renormalized operators with insertions of factors of the bare current J are independent of the cutoff to all orders in perturbation theory, when expressed in terms of the renormalized coupling. We will use this result, which can be proved[3] using the Ward identities for conserved currents, throughout the paper.

By dimensional analysis the current–current correlation function $\langle J^\mu(x) J^\nu(0) \rangle$ has dimension $[\text{length}]^{-2d+2}$. The limit in which the length l of an oriented SAW tends to infinity corresponds to the renormalized mass m of the continuum complex $O(N)$ model tending to zero. In the massless regime, $\langle J^\mu(x) J^\nu(0) \rangle$ must scale with $|x|$ like $|x|^{-2d+2}(\kappa |x|)^y$, where κ is the usual momentum scale which must be introduced in order to define a renormalized massless field theory, and y is an "anomalous" dimension. Because J does not acquire an anomalous dimension, $y = 0$. Thus, in the massless theory $\langle J^\mu(x) J^\nu(0) \rangle$ must decay like $|x|^{-2d+2}$.

Furthermore, by rotational covariance the correlation functions must be proportional to $x^\mu x^\nu + a x^2 \delta^{\mu\nu}$ times a scalar, and by current conservation, $\partial_\mu J^\mu = 0$, $a = -1/2$. So

$$\langle J^\mu(x)\, J^\nu(0) \rangle = k(d) \frac{(x^\mu x^\nu - \frac{1}{2} x^2\, \delta^{\mu\nu})}{|x|^{2d}} \tag{11}$$

where the amplitude $k(d)$ is a dimensionless function of spatial dimension d.

Since the form of the current–current correlations are completely determined by the above considerations, we have only to calculate the dimensionless amplitude $k(d)$. For long oriented random walks, which are described by the free theory, $g = 0$, this is easy: we can compute the current–current correlation functions exactly using Wick's theorem[7] and then read off the amplitude $k_G(d)$. In the free massless theory, [Eq. (10)] with $g = 0$, the two-point function is simply

$$\langle S_i(0)\, S_j^*(x) \rangle \equiv G(x) = \delta_{i,j} \frac{\Gamma(d/2 - 1)}{4\pi^{d/2} |x|^{d-2}} \tag{12}$$

and Wick's theorem gives us

$$\langle J^\mu(x)\, J^\nu(0) \rangle = 2\lambda^2 [G(x)\, \partial^\nu \partial^\mu G(x) - \partial^\mu G(x)\, \partial^\nu G(x)]$$

$$= k_G(d) \frac{(x^\mu x^\nu - \frac{1}{2} x^2\, \delta^{\mu\nu})}{|x|^{2d}} \tag{13}$$

where

$$k_G(d) = \frac{\lambda^2 \Gamma^2(d/2)}{(d-2)\pi^d} = \frac{4\lambda^2}{(d-2)\, S(d)^2} \tag{14}$$

and $S(d)$ is the area of a $(d-1)$-sphere. Note that the functional form [Eq. (13)] of the current–current correlation functions evaluated at the critical point of the free theory agrees with the general form given in Eq. (11), which is valid at all critical points of the complex $O(N)$ model.

The amplitude $k(d)$ of an oriented SAW equals the amplitude of an oriented random walk $k_G(d)$ in dimensions $d \geq 4$. The upper critical dimension of a SAW is four because the Hausdorff dimension of a random walk is two. In four dimensions, two-dimensional surfaces intersect generically only at isolated points, and in higher dimensions they do not intersect at all, except in exceptional cases. Thus, in dimensions $d \geq 4$, a random walk is, *de facto*, a SAW.[6] Below four dimensions, $k(d)$ is modified by the interaction.

From Eq. (11) we see that correlations in direction fall off like $1/|x|^{2d-2}$. In order to see what the correlation function tells us about angular correlations, imagine fixing the tangent vector at the origin $J(0)$ to be in some particular direction. Then the correlation function $\langle J^\mu(x) J^\nu(0) \rangle$ equals twice the average value of a tangent vector at x, given that the SAW goes through the origin with tangent vector $J(0)$. Azimuthal symmetry around the line through $J(0)$ allows us to restrict our attention to any one of the planes to which $J(0)$ belongs [azimuthal symmetry implies that, on average, components of $J(x)$ out of such a plane are zero]. Taking the vector at the origin $J(0)$ to lie on the x^1 axis and the plane to be the (x^1, x^2) plane, and introducing polar coordinates $x^1 = |x| \cos \theta$, $x^2 = |x| \sin \theta$, we find the average values of the x^1 and x^2 components of $J(x)$ to be

$$\langle J^1(x) \rangle = \frac{k \cos 2\theta}{4 |x|^{2d-2}} \tag{15}$$

$$\langle J^2(x) \rangle = \frac{k \sin 2\theta}{4 |x|^{2d-2}} \tag{16}$$

So we see that the average field of tangent vectors of an oriented SAW, with a fixed tangent vector at the origin, looks something like the field of an electric dipole around $J(0)$.

In two dimensions it is convenient to introduce complex coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. In the coordinate basis of z and $\bar{z}$ the metric becomes $ds^2 = dz\, d\bar{z}$ and current conservation reads

$$\partial_\mu J^\mu = \partial_z J^z + \partial_{\bar{z}} j^{\bar{z}} = 2(\partial_z \bar{J} + \partial_{\bar{z}} J) = 0 \tag{17}$$

where $J = J_z$ and $\bar{J} = J_{\bar{z}}$. In terms of J, $\bar{J}$, z, and $\bar{z}$ the direction–direction correlation functions [Eq. (11)] are easily found to be

$$\langle J(z) J(0) \rangle = \frac{k(2)/4}{z^2} \tag{18}$$

$$(\bar{J}(\bar{z}) \bar{J}(0) \rangle = \frac{k(2)/4}{\bar{z}^2} \tag{19}$$

$$\langle J(z) \bar{J}(0) \rangle = 0 \tag{20}$$

From this we see that J is a function of z only and $\bar{J}$ a function of $\bar{z}$, so that J and $\bar{J}$ are conserved separately in Eq. (17). This is not surprising, because the complex $O(N)$ model at a critical point, which is where the functional form Eq. (11) is valid, is conformally invariant. From conformal

field theory[12] we know that every operator of a two-dimensional conformally invariant theory is characterized by a pair of real numbers h and $\bar{h}$, called conformal dimensions, which determine how the operator transforms under conformal transformations. In particular, the two-point function of an operator A with conformal dimensions $(h, \bar{h})$ is constrained by scale invariance to be of the form

$$\langle A(z, \bar{z}) \, A(0) \rangle = \frac{a}{z^{2h} \bar{z}^{2\bar{h}}} \tag{21}$$

where a is a constant. The conformal dimensions are related to the usual scaling dimension x and the spin s by $x = h + \bar{h}$ and $s = h - \bar{h}$. Since J is a vector, we know that it has spin one. Therefore $s = 1 = h - \bar{h}$. On the other hand, since J is a conserved current, its scaling dimension is not changed by the interactions, and is given by dimensional analysis to be $x = 1 = h + \bar{h}$. Thus, the conformal dimensions of the operator J are $h = 1$ and $\bar{h} = 0$, while those of $\bar{J}$ are $h = 0$ and $\bar{h} = 1$, which is just what we found above.

4. NORMALIZATION OF J

The normalization of J is fixed by demanding

$$\langle S_1(0) \, S_1^*(a) \rangle = \int ds_\mu \, \langle J^\mu(r) \, S_1(0) \, S_1^*(a) \rangle \tag{22}$$

where the integral is over a $(d-1)$-sphere around 0, which does not contain the point a. This amounts to taking the charge q of the spin S_1 to equal one. In terms of an oriented polymer, Eq. (22) sets the magnitude $|J|$ of the current J (really a current density) equal to A^{-1}, where A is the cross-sectional "area" (dimension $[\text{length}]^{d-1}$) of the polymer. (The cross-sectional "area" of a polymer is determined by the distance at which excluded-volume effects become important.) One may also regard one end of the polymer as a source, and the other end a sink of electric charge. Then Eq. (22) corresponds to unit current $I = |J| A = 1$ flowing through the polymer.

Since J is not renormalized, corrections to the free theory from the interaction are the same on both sides of Eq. (22), and thus cancel. We show this explicitly in Appendix B to second order in the ε-expansion. We can therefore use the free theory to calculate the constant λ. Using Wick's theorem to calculate the correlation function on the right-hand side of Eq. (22), we have

$$\frac{\Gamma(d/2-1)}{4\pi^{d/2}a^{d-2}} = \lambda \frac{\Gamma^2(d/2-1)r^{d-1}}{16\pi^d} \int d\Omega_d \left[G(r)\, \partial_r G(r-a) - G(r-a)\partial_r G(r) \right]$$

$$= \lambda \frac{\Gamma^2(d/2-1)r^{d-1}}{16\pi^d} \left(\frac{(2\pi)(\pi)^{(d-3)/2}}{\Gamma((d-1)/2)} \right) \left(\frac{(d-2)\,\Gamma((d-1)/2)\,\Gamma(1/2)}{a^{d-2}\Gamma(d/2)} \right)$$

$$= \lambda \frac{\Gamma(d/2-1)}{4\pi^{d/2}a^{d-2}} \tag{23}$$

Thus, $\lambda = 1$ (which is just what we would have gotten from Noether's theorem).

In two dimensions, Eq. (22) implies that the operator product expansion[13,14] of $J(z)$ with $S(0)$ is of the form

$$J(z)\,S(0) = \frac{\alpha}{z}\,S(0) + \cdots \tag{24}$$

with α a constant. There is a similar equation for S^* with $-\alpha$ instead of α. Again, the constant α is determined by Eq. (22),

$$\langle S(r_1)\,S^*(r_2) \rangle = \int ds_\mu \, \langle J^\mu(r)\, S(r_1)\, S^*(r_2) \rangle \tag{25}$$

where the integral is over the boundary of a region which contains the point r_1 but not the point r_2. In complex coordinates the integral becomes

$$\oint \frac{dz}{i} \, \langle J(z)\, S(z_1,\bar{z}_1)\, S^*(z_2,\bar{z}_2) \rangle - \oint \frac{d\bar{z}}{i} \, \langle \bar{J}(\bar{z})\, S(z_1,\bar{z}_1)\, S^*(z_2,\bar{z}_2) \rangle \tag{26}$$

This in turn equals

$$\langle S(r_1)\,S^*(r_2) \rangle \left[\oint \frac{dz}{i} \left(\frac{\alpha}{z-z_1} - \frac{\alpha}{z-z_2} \right) - \oint \frac{d\bar{z}}{i} \left(\frac{-\alpha}{\bar{z}-\bar{z}_1} + \frac{\alpha}{\bar{z}-\bar{z}_2} \right) \right] \tag{27}$$

Since the contour only encloses the point r_1, we have, after doing the contour integrals,

$$\alpha = \frac{1}{4\pi} \tag{28}$$

So that

$$J(z)\,S(0) = \frac{1}{4\pi z}\,S(0) + \cdots \tag{29}$$

5. CALCULATION OF $k(2)$

In order to calculate $k(2)$ we shall map the problem onto a Coulomb gas. The mapping is well known,[4,5] but will be sketched briefly so that it is clear which operator in the Coulomb gas corresponds to the tangent vector J. There are essentially two steps. First the $O(N)$ model is mapped onto a discrete solid-on-solid (SOS) model. Then it is argued that the SOS model renormalizes onto a Gaussian model. The spins of the $O(N)$ model spin wave-vortex operators in the Gaussian model, and the current J becomes the curl of the bosonic field ϕ.

One starts with a complex $O(N)$ model on a honeycomb lattice with N-component complex spins and partition function

$$Z = \int \prod_r d\Omega(r) \prod_{\langle r, r' \rangle} [1 + \beta \mathbf{S}(\mathbf{r}) \cdot \mathbf{S}^*(\mathbf{r}') + \text{c.c.}] \tag{30}$$

The first product is over all the site of the lattice, the second is over all nearest neighbor pairs of spins. The spins are normalized so that $\mathbf{S} \cdot \mathbf{S}^* = N$, and $d\Omega$ is the normalized measure of a $(2N-1)$-sphere, $\int d\Omega = 1$. With these normalizations, $\int d\Omega(S_i S_j^*) = \delta_{i,j}$. Note also that later formulas involving N will differ by a factor of two from those of Neinhuis because the complex $O(N)$ model is a real $O(2N)$ model.

Since the coordination number of a honeycomb lattice is three, the only graphs which contribute to the high-temperature expansion are oriented loops. Thus

$$Z = \sum_{\substack{\text{oriented} \\ \text{loops}}} \beta^l N^c = \sum_{\text{loops}} \beta^l (2N)^c \tag{31}$$

where c is the total number of oriented loops in a graph, and l is the sum of their lengths.

Each oriented graph in the high-temperature expansion can be associated with a configuration of a discrete SOS model on the triangular dual lattice. One assigns a height variable $\phi(r)$, taken to be an integral multiple of π, to each site r of the dual lattice in such a way that the height on either side an oriented bond differs by π, but is otherwise constant. We shall take the higher side to be on the left of the oriented bond. The total number of bonds in the original graph is given in terms of the variables ϕ by $(1/\pi) \sum_{\langle r, r' \rangle} |\phi(r) - \phi(r')|$, so that the sum over configurations of the height variables (restricted so that adjacent heights differ by at most π), with each configuration weighted by $\beta^{(1/\pi) \sum_{\langle r, r' \rangle} |\phi(r) - \phi(r')|}$, gives the complex $O(N)$ partition function, except that each oriented loop is weighted by 1 rather than N. This is easily remedied by assigning to each left turn of an

oriented loop the weight e^{iu} and each right turn the weight e^{-iu}. On the plane, the total number of right turns of an oriented loop minus the total number of left turns equals ± 6, so summing over both orientations of the loop, we see that if we set

$$2 \cos 6u = 2N \tag{32}$$

the partition functions of the complex $O(N)$ model and the TSOS model are equal.

Based on an analysis of renormalization group flows, Nienhuis argued that the critical triangular SOS (TSOS) model renormalizes onto the Gaussian model

$$A = \frac{g}{4\pi} \int d^2r\, \partial_\mu \phi\, \partial^\mu \phi \tag{33}$$

with coupling constant g given by

$$2N = -2 \cos \pi g, \qquad g \in [1, 2] \tag{34}$$

For $N = 0$, $g = 3/2$. However, this action as it stands only describes the critical complex $O(1)$ model, i.e., the XY model. In order to describe an $O(N)$ model for arbitrary N, it is necessary to add to the action $2ie\phi(\infty)$, which corresponds to putting an electric charge at infinity (see, e.g., ref. 15).

To see this, consider the spin–spin correlation function $\langle S_1(0)\, S_1^*(r) \rangle$ of the complex $O(N)$ model. On the honeycomb lattice

$$\langle S_1(0)\, S_1^*(r) \rangle = \sum_{\substack{\text{oriented} \\ \text{SAWs}}} \beta^l \tag{35}$$

where the sum is over all oriented SAWs from 0 to r, and l is the length of a walk. In general there are also loops, but these are suppressed when $N = 0$. In the TSOS model an oriented SAW represents a domain wall of height π, with a vortex operator at one extremity and an antivortex operator at the other (Fig. 4). Since the discontinuity in the height ϕ at the domain wall equals π, the vortex (antivortex) operator must carry magnetic charge $m = 1/2$ $(-1/2)$. We cannot, however, identify the spin operator S with a vortex operator, because in the $O(N)$ model SAWs of equal length are weighted equally, while in the TSOS model a walk picks up a factor of $\exp(\pm 6iu)$ each time it winds around one of the endpoints. This factor can be accounted for by multiplying both vortex operators by a spin wave operator
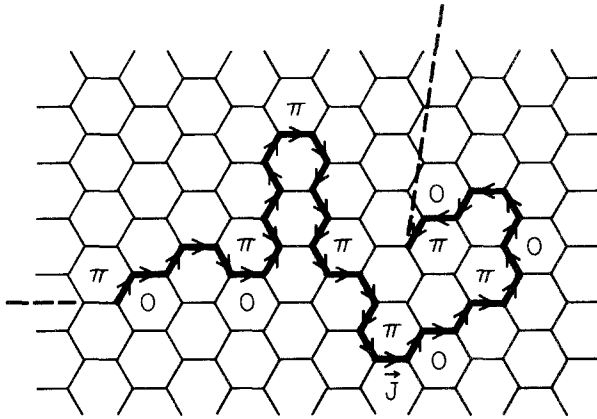
$$\exp(-6iu\phi/\pi) \tag{36}$$

Fig. 4.   A graph which contributes to $\langle S_1(0)\, S_1^*(r)\rangle$ on the honeycomb lattice. In the TSOS
model the walk is represented by a wall of height π.

for, each time a walk winds around an endpoint, the height of the endpoint
ϕ changes by $\pm\pi$, which exactly compensates for the factor from the
curvature of the walk. So the spins S, S^* are represented by a combined
spin-wave vortex operator $O_{e,m}$ with electric charge $e = -6u/\pi = 1 - g$ and
magnetic charge $m = \pm 1/2$. The excess charge $2e$ in the correlation function
is exactly canceled by the charge at infinity in the action.

The magnetic operator $O_{0,m}$ has no simple representation in terms of
ϕ, but it can be written as an exponential the variable ψ dual[16] to ϕ. Thus,
in the continuum, the spin operator of the $O(N)$ model has the
representation

$$O_{e,m} = e^{ie\phi + mg\psi} \tag{37}$$

with e, m, and g given above. ϕ and ψ are related by the Cauchy–Riemann
equations

$$\partial^\mu\psi = \varepsilon^{\mu\nu}\partial_\nu\phi \tag{38}$$

where $\varepsilon^{\mu\nu}$ is the two-dimensional totally antisymmetric tensor. These equa-
tions imply that the function $\theta = \frac{1}{2}(\phi + i\psi)$ is holomorphic.

Since SAWs are domain walls in the TSOS model, it is clear that the
magnitude of a tangent vector J of a SAW is proportional to the difference
in height between adjacent hexagons $\phi(r) - \phi(r')$, and that the direction is
along the common face of the hexagons, perpendicular to the discrete
gradient. In the continuum limit the discrete difference becomes a gradient,
so that

$$J^\mu = \bar\lambda\varepsilon^{\mu\nu}\partial_\nu\phi = \bar\lambda\partial^\mu\psi \tag{39}$$

where $\bar\lambda$ is a constant.

The constant $\bar{\lambda}$ is computed by calculating the operator product expansion

$$J_z(z)\, O_{e,m}(0) = \bar{\lambda}\partial_z\psi O_{e,m}(0) \tag{40}$$

and comparing the result with Eq. (29). To do this we write ϕ and ψ in terms of the holomorphic function θ and its complex conjugate $\bar{\theta}$: $\phi = \theta(z) + \bar{\theta}(\bar{z})$ and $\psi = (1/i)[\theta(z) - \bar{\theta}(\bar{z})]$. Correlation functions are given by the inverse of the quadratic part of the action, Eq. (33),

$$\langle \phi(r)\, \phi(0) \rangle = \frac{-1}{g}\ln|r| \tag{41}$$

so that

$$\langle \theta(z)\, \theta(0) \rangle = \frac{-1}{2g}\ln z$$
$$\langle \theta(z)\, \bar{\theta}(0) \rangle = 0 \tag{42}$$

The operator product expansion of $J(z)$ with $O_{e,m}(0)$ can be computed using Wick's theorem. Since $\langle \theta\bar{\theta} \rangle$ is identically zero, $\bar{\theta}$ does not play a role in the contractions. So

$$J(z)\, O_{e,m}(0) = \bar{\lambda}\partial_z\psi e^{ie\phi + mg\psi}$$

$$= \frac{\bar{\lambda}}{i}\, e^{(ie + img)\theta}\partial_z\theta \sum_{n=0}^{\infty} \frac{(ie - img)^n \theta^n}{n!}$$

$$= \frac{\bar{\lambda}}{i}\, e^{(ie + img)\theta}\partial_z \sum_{n=1}^{\infty} \frac{(ie - img)^n}{n!}\left(\frac{-1}{2g}\ln z\right)n\theta^{n-1}$$

$$= \bar{\lambda}\left(\frac{mg - e}{2gz}\right)O_{e,m}(0) \tag{43}$$

On the other hand, from Eq. (29) we have

$$J(z)\, O_{e,m}(0) = \frac{1}{4\pi z}\, O_{e,m} \tag{44}$$

from which we conclude

$$\bar{\lambda} = \frac{g}{2\pi(e - mg)} \tag{45}$$

Knowing $J(z)$, we can compute $k(2)$,

$$\langle J(z)\, J(0)\rangle = \bar{\lambda}^2 \left\langle \frac{1}{i}\, \partial_z \theta(z) \frac{1}{i}\, \partial_z \theta(0) \right\rangle = \frac{\bar{\lambda}^2}{2g}\frac{1}{z^2} = \frac{k(2)}{4}\frac{1}{z^2} \qquad (46)$$

Consequently,

$$k = \frac{2\bar{\lambda}^2}{g} = \frac{g}{2\pi^2(e-mg)^2} \qquad (47)$$

Setting $g = 3/2$, $e = 1 - g = -1/2$, and $m = 1/2$, we have our result

$$k(2) = \frac{12}{25\pi^2} \qquad (48)$$

6. ϵ-EXPANSION OF $k(d)$

We now turn to the ε-expansion[8,9] of $k(d)$. We shall take $2\varepsilon = 4 - d$. This differs by a factor of two from the usual choice of ε in the statistical mechanics literature, but it is common in particle physics calculations, and involves fewer factors of $1/2$ in the gamma functions which always arise. Since the conserved current J is not renormalized, correlation functions of renormalized operators with insertions of the bare J (composed of the bare fields: $J = J_R = \phi^* \partial \phi - \phi \partial \phi^*$) are finite when expressed in terms of the renormalized coupling. Of course, the two-point function $\langle J^\mu(x)\, J^\nu(0)\rangle$ diverges as x approaches 0. In momentum space this is manifested by a pole in ε. However, for the purpose of calculating $k(d)$, we can work in position space and keep x finite, so this is not a problem.

The first step in the calculation is to write down the diagrams which contribute to $\langle J^\mu(r)\, J^\nu(0)\rangle$. To second order in g, these are shown in Fig. 5. Note that there are no tadpole diagrams. This is because such
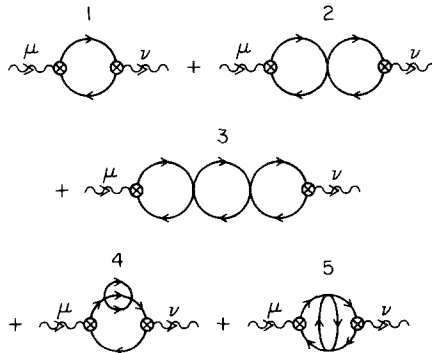

Fig. 5. Feynman diagrams which contribute to $\langle J^\mu(x)\, J^\nu(0)\rangle$, to second order in g.

diagrams, when evaluated in momentum space, have no external momenta flowing through them. The argument of the integral they represent is therefore a pure power (the theory is massless), and in dimensional regularization integrals of powers are defined to be zero.[17]

The first diagram is just the free field correlation function. In position space it equals

$$\frac{4}{(d-2)\,S(d)^2}\,\frac{(r^\mu r^\nu - \frac{1}{2}r^2\delta^{\mu\nu})}{|r|^{2d}} \tag{49}$$

and in momentum space,

$$\int \frac{d^d p}{2\pi^d}\,\frac{(2p^\mu - k^\mu)(2p^\nu - k^\nu)}{p^2(p-k)^2} = -\frac{\Gamma(\varepsilon)\,\Gamma^2(1-\varepsilon)(k^\mu k^\nu - \delta^{\mu\nu}k^2)}{(4\pi)^{d/2}(d-1)\,\Gamma(2-2\varepsilon)(k^2)^\varepsilon} \tag{50}$$

Since we are only interested in the amplitude $k(d)$, we can trace over the vector indices. This will make the evaluation of diagram 5 easier. The trace of Eq. (50) is just

$$\frac{\Gamma(\varepsilon)\,\Gamma^2(1-\varepsilon)(k^2)^{1-\varepsilon}}{(4\pi)^{d/2}\,\Gamma(2-2\varepsilon)} \tag{51}$$

The second and third diagrams are identically equal to zero. This is because the current J consists of two terms with opposite signs, so that the contribution to diagrams 2 and 3 from $\langle \phi^*\partial^\mu\phi(r)\,\phi^*\partial^\nu\phi(0)\rangle$ is canceled by the contribution from $\langle \phi^*\partial^\mu\phi(r)\,\phi\partial^\nu\phi^*(0)\rangle$. So the first corrections to the free theory are of order g^2.

The fourth and fifth diagrams will be evaluated in Appendix A.[18] Two aspects of their calculation might be noted. First, since we are calculating to second order in g_R^*, or to second order in ε, it is sufficient to calculate both diagrams to order $1/\varepsilon$ in momentum space. Why order $1/\varepsilon$ rather than order 1? Because in momentum space we pick up a pole in ε from the high-momentum region, which corresponds in position space to $r \to 0$. When we Fourier transform back to position space this pole disappears. Second, diagrams 4 and 5 each have double poles in ε. It is only when we add them together that the double poles cancel, and we are left with a simple pole.

From the Appendix, the sum of the traces of diagrams 1, 4, and 5 equals

$$\frac{\Gamma(\varepsilon)\,\Gamma^2(1-\varepsilon)(k^2)^{1-\varepsilon}}{(4\pi)^{d/2}\,\Gamma(2-2\varepsilon)}\left(1 - \frac{g^2}{8(4\pi)^d}\right) \tag{52}$$

Comparing this expression with Eq. (51), we see that to second order in g,

$$\langle J^\mu(r) \, J^\nu(0) \rangle = \left(1 - \frac{g^2}{8(4\pi)^d}\right) \frac{4}{(d-2)\,S(d)^2} \frac{(r^\mu r^\nu - \frac{1}{2}r^2 \delta^{\mu\nu})}{|r|^{2d}} \tag{53}$$

The fixed point at which we will evaluate the above expression is given by the zero of the beta function for the renormalized coupling constant g_R. To first order in ε it equals

$$\frac{g_R^*}{(4\pi)^d} = \frac{\varepsilon}{2} \tag{54}$$

so that

$$\langle J^\mu(r) \, J^\nu(0) \rangle = \left(1 - \frac{\varepsilon^2}{32}\right) \frac{4}{(d-2)\,S(d)^2} \frac{(r^\mu r^\nu - \frac{1}{2}r^2 \delta^{\mu\nu})}{|r|^{2d}} \tag{55}$$

Expanding $[(d-2)\,S(d)^2]^{-1}$ (see ref. 19) in powers of ε to order ε^2 and multiplying by $(1 - \varepsilon^2/32)$ gives us the ε-expansion,

$$k_\varepsilon(\varepsilon) = \frac{1}{2\pi^4} \left\{ 1 + (2\gamma - 1 + 2\ln\pi)\varepsilon \right.$$

$$\left. + \frac{1}{2}\left[\gamma(\gamma-1) + \frac{\pi^2}{6} + (2\gamma-1)\ln\pi + (\ln\pi)^2 - \frac{1}{16}\right]\varepsilon^2 \right\} + \cdots$$

$$\cong 0.0051330 + 0.0125445\varepsilon + 0.0210452\varepsilon^2 + \cdots \tag{56}$$

where γ is Euler's constant $= 0.5772...$, and the subscript on k_ε means that the function is to be evaluated with ε rather than d.

In two dimensions ($\varepsilon = 1$) the ε-expansion predicts $k(2) \cong 0.0387$, which is smaller than the exact value $k(2) = 12/25\pi^2 \cong 0.0486$ by about 20%. In three dimensions, the expansion gives $k(3) \cong 0.0167$. Knowledge of the exact value of $k(2)$ in two dimensions should allow us to improve this estimate. [It fact we also know the exact $k(1)$ in one dimension: $k(1) = 4$, because there are only two oriented SAWs which contribute the current–current correlation function. However, this does not help us improve our estimate of $k(3)$, because $k(1)$ is about 100 times larger than $k(2)$, so that it is difficult to incorporate the point $k(1)$ with a simple Padé form.] In particular, we can fit the ε-expansion to a Padé approximant[20] of the form

$$k_\varepsilon(\varepsilon) = \frac{k_\varepsilon(1) + (1-\varepsilon)\,P(\varepsilon)}{1 + (1-\varepsilon)\,Q(\varepsilon)} \tag{57}$$

where P and Q are polynomials in ε. Since we only know the first three terms in the ε-expansion, we can fit three coefficients in the Padé approximant. The case where P is a second-order polynomial and $Q = 0$, and where P is linear and Q a constant, have almost the same graph. In the first case we find $k(3) \cong 0.01767$ and in the second, $k(3) \cong 0.01800$. The other Padé approximants, with P a constant and with P equal to zero, cannot be fit to the ε-expansion. Thus we conjecture that $k(3) \cong 0.01783$. The error in this estimate cannot be determined without going to higher order in the ε-expansion. However, we expect that (1) the ε-expansion will be more accurate in three dimensions than in two, and (2) the Padé approximant, which incorporates the exact value of $k(2)$ in two dimensions, makes the agreement even better. Since the ε-expansion alone is only about 20% too small in two dimensions, we should expect that the ε-expansion, with the exact two-dimensional result taken into account by the Padé approximant, gives a value of $k(3)$ in three dimensions to within a few percent of the exact value. So we estimate that

$$k(3) = 0.0178 \pm 0.0005 \tag{58}$$

where the subjective error reflects the above considerations.

APPENDIX A

1. In this Appendix we compute diagrams 4 and 5 of Fig. 5. Diagram 4 represents the integral

$$-\frac{g^2}{2}(N+1) \int \frac{d^d l}{(2\pi)^d} \frac{(2l^\mu - k^\mu)(2l^\nu - k^\nu)\,\Sigma(l^2)}{l^4(l-k)^2} \tag{A.1}$$

where $\Sigma(l^2)$ is the lowest order contribution to the self-energy, and is given by

$$\Sigma(l^2) = \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{p^2 q^2 (l - p - q)^2} = \frac{\Gamma^3(1-\varepsilon)\,\Gamma(-1+2\varepsilon)}{(4\pi)^d \Gamma(3-3\varepsilon)} (l^2)^{1-2\varepsilon} \tag{A.2}$$

The integrals in Eq. (A.1) are straightforward. I find, after taking the trace, and setting $N = 0$, that the diagram equals

$$\frac{g^2}{2} \left[\frac{\Gamma(\varepsilon)\,\Gamma(1-\varepsilon)^2 (k^2)^{1-\varepsilon}}{(4\pi)^{d/2}\,\Gamma(2-2\varepsilon)} \right]$$

$$\times \left[\frac{\Gamma(1-\varepsilon)^2 \Gamma(-1+2\varepsilon)\,\Gamma(3\varepsilon)\,\Gamma(1-3\varepsilon)\,\Gamma(2-2\varepsilon)[1+2\varepsilon+o(\varepsilon^2)]}{(4\pi)^d\,\Gamma(\varepsilon)\,\Gamma(3-3\varepsilon)\,\Gamma(1+2\varepsilon)\,\Gamma(2-4\varepsilon)(k^2)^{2\varepsilon}} \right] \tag{A.3}$$

Notice that there is a pole of order ε^2, which must be canceled by diagram 5.

Diagram 5 represents the integral

$$
\frac{-g^2}{4}(N+1)\int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\frac{d^d r}{(2\pi)^d}\frac{(2p^\mu - k^\mu)(2q^\nu - k^\nu)}{p^2 q^2 (p-k)^2 (q-k)^2 r^2 (p-q-r)^2}\quad .\text{(A.4)}
$$

The internal loop is easily integrated. Doing the integral, taking the trace, and setting $N=0$, one has

$$
\frac{g^2}{2}\frac{\Gamma(\varepsilon)\,\Gamma(1-\varepsilon)^2}{(4\pi)^d\,\Gamma(2-2\varepsilon)}\int \frac{d^d p}{(2\pi)^d}\frac{d^d q}{(2\pi)^d}\frac{(p-q)^2 + k^2/2 - 2p^2}{p^2 q^2 (p-k)^2 (q-k)^2 (p-q)^{2\varepsilon}}\qquad \text{(A.5)}
$$

There are now three integrals to do, In the third integral, the p^2 in the numerator cancels the similar term in the denominator, and the integral equals

$$
-g^2 \frac{\Gamma(1-\varepsilon)^4\,\Gamma(-1+3\varepsilon)(k^2)^{1-3\varepsilon}}{(4\pi)^{3d/2}(-1+2\varepsilon)(2-3\varepsilon)\,\Gamma(3-4\varepsilon)}\qquad \text{(A.6)}
$$

The remaining two integrals of diagram 5 are more difficult, but can be evaluated to the desired order by expanding the integrals in position space with Gegenbauer polynomials. Using the formulas of Chetyrkin et al.,[18] we find that sum of the first and second integrals of Eq. (A.5) equals

$$
\frac{g^2 \Gamma(\varepsilon)\,\Gamma(1-\varepsilon)^2 (k^2)^{1-\varepsilon}}{(4\pi)^{d/2}\,\Gamma(2-2\varepsilon)}\left[\frac{\Gamma(1-\varepsilon)^3\,\Gamma(2-2\varepsilon)\,\Gamma(3\varepsilon)}{8(4\pi)^d\,\Gamma(\varepsilon)\,\Gamma(2-\varepsilon)\,\Gamma(1-4\varepsilon)(k^2)^{2\varepsilon}}\right]
$$
$$
\times\left[\frac{1}{\varepsilon}+\frac{21}{2}+o(\varepsilon)\right]\qquad \text{(A.7)}
$$

Adding all the contributions from diagram 4 and diagram 5 gives

$$
\frac{\Gamma(\varepsilon)\,\Gamma(1-\varepsilon)^2 (k^2)^{1-\varepsilon}}{(4\pi)^{d/2}\,\Gamma(2-2\varepsilon)}\left(-\frac{g^2}{8(4\pi)^d}\right)\qquad \text{(A.8)}
$$

APPENDIX B

In this Appendix we show explicitly (to second order in ε) that Eq. (22) is unchanged by the interactions. We shall first compute the renormalized correlation function

$$
\langle \phi_R(p)\,\phi_R^*(-p)\rangle = Z_\phi^{-1}\langle \phi(p)\,\phi^*(-p)\rangle\qquad \text{(B.1)}
$$

The wave function renormalization constant Z_ϕ is chosen in such a way that it cancels the poles in ε which arise in the perturbative calculation of the bare correlation function. We will use the minimal subtraction scheme. To second order in g only two diagrams contribute to $\langle \phi(p)\, \phi^*(-p) \rangle$: the free field correlation function, and the free field correlation function with one insertion of the self-energy $\Sigma(p^2)$, Eq. (A.2). So

$$\langle \phi(p)\, \phi^*(-p) \rangle = \left[\frac{1}{p^2} + \frac{g^2}{2}\frac{1}{p^2}\, \Sigma(p^2)\frac{1}{p^2} \right]$$

$$= \frac{1}{p^2}\left\{ 1 - \frac{g^2}{8\varepsilon(4\pi)^d}\left[1 + \bar\gamma\varepsilon + \frac{5\varepsilon}{2} + o(\varepsilon^2) \right] \right\} \qquad (\text{B.2})$$

where $\bar\gamma = \ln(4\pi) + 2 - \gamma$ and γ is Euler's constant. We therefore have from Eq. (B.2)

$$Z_\phi = 1 - \frac{g^2}{8\varepsilon(4\pi)^d} \qquad (\text{B.3})$$

and

$$\langle \phi_R(p)\, \phi_R^*(-p) \rangle = \frac{1}{p^2}\left\{ 1 - \frac{g^2}{8(4\pi)^d}\left[\frac{5}{2} + \bar\gamma + o(\varepsilon) \right] \right\} \qquad (\text{B.4})$$

Now we turn to the right-hand side of Eq. (22). To second order in g, three diagrams contribute. These are shown in Fig. 6. The first of these diagrams equals

$$i(2\pi)^d\, \delta(k + p + q)\, G(p)\, G(q)(q^\mu - p^\mu) \qquad (\text{B.5})$$
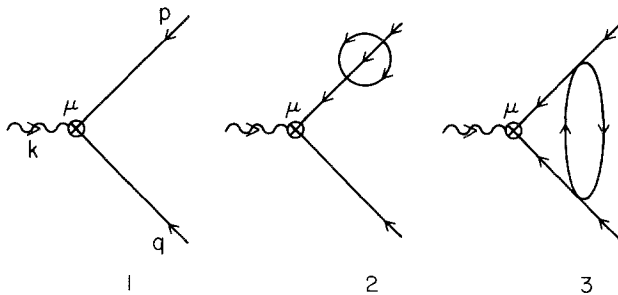

Fig. 6.   The nonzero Feynman diagrams which contribute to $\langle J^\mu(k)\, \phi(p)\, \phi^*(q) \rangle$ to second order in g.

where $G(p) = 1/p^2$. The second diagram equals

$$i(2\pi)^d\,\delta(k+p+q)\,G(p)\,G(q)(q^\mu - p^\mu)\frac{g^2}{2}\,[G(p^2)\,\Sigma(p^2) + G(q^2)\,\Sigma(q^2)]$$

$$= i(2\pi)^d\,\delta(k+p+q)\,G(p)\,G(q)(q^\mu - p^\mu)\frac{g^2}{(4\pi)^d}$$

$$\times \left[\frac{\Gamma(1-\varepsilon)^3\,\Gamma(-1+2\varepsilon)[1+o(\varepsilon)]}{\Gamma(3-3\varepsilon)}\right] \qquad\qquad (\text{B.6})$$

and the third diagram equals

$$i(2\pi)^d\,\delta(k+p+q)\,G(p)\,G(q)\frac{g^2}{2}\int\frac{d^d s}{(2\pi)^d}\frac{d^d l}{(2\pi)^d}\frac{(2s^\mu - k^\mu)}{s^2(k-s)^2\,l^2(l-s-k)^2}$$

$$= i(2\pi)^d\,\delta(k+p+q)\,G(p)\,G(q)(q^\mu - p^\mu)\frac{g^2}{4(4\pi)^d}$$

$$\times \left[\frac{\Gamma(1-\varepsilon)^3\,\Gamma(2\varepsilon)[1+7\varepsilon/2 + o(\varepsilon^2)]}{\Gamma(2-3\varepsilon)}\right] \qquad\qquad (\text{B.7})$$

Adding the three contributions above and multiplying by Z_ϕ^{-1}, we have to second order in ε

$$\langle J^\mu(k)\,\phi_R(p)\,\phi_R^*(q)\rangle$$

$$= i(2\pi)^d\,\delta(k+p+q)\,G(p)\,G(q)(q^\mu - p^\nu)\left\{1 - \frac{g^2}{8(4\pi)^d}\left[\frac{5}{2} + \bar\gamma + o(\varepsilon)\right]\right\}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{B.8})$$

Comparing this with Eq. (B.4), we see that the corrections to the free theory from the interaction cancel on either side of Eq. (22), as was to be shown.

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